SOUND INSULATION BY A FINITE CYLINDER

by

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Introduction.

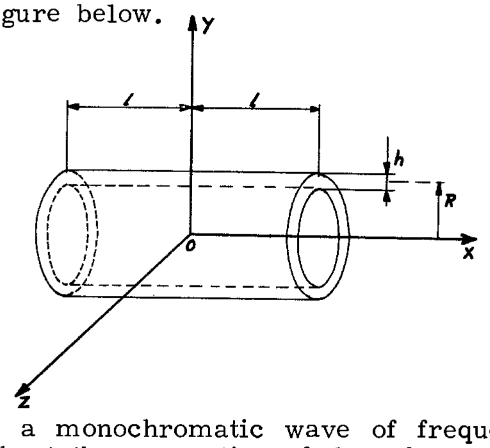
This paper presents an analysis of the sound field generated by a simple source which is surrounded by a flexible cylindrical tube of finite length. The acoustic properties of the material of the tube wall differ only slightly from those of the surrounding medium. Assuming the thickness of the tube wall small compared to the wave length of sound a perturbation method, often referred to as the "Born approximation", can be used. In this approximation the whole scattering field is considered as a perturbation modifying the primary wave motion of the simple source at the origin $\varphi_s = r^{-1} \cdot e^{i(kr-\omega t)}$.

The first Born approximation involves substituting the unperturbed wave function under the integral sign in the integral equation for scattering. The second approximation is obtained by substituting the first approximation for the unknown function and so on. An asymptotic solution is obtained for the sound pressure at large distances from the scattering region.

Statement of the problem.

A zero order spherical sound source supplying a constant volume velocity is located at the origin of a rectangular coordinate system x-y-z.

A circular cylindrical tube of length 2l, whose axis coincides with the x-axis, encloses the sound source at its center. The wall thickness of the tube is h, and the mean radius between outer- and inner cylinder surface is denoted by R. See figure below.



The source radiates a monochromatic wave of frequency $\omega/2\pi$. The following assumptions about the properties of the tube are made:

The flexibility of the tube wall is such, that Young's modulus of elasticity can be neglected.

The wall thickness h is small compared to the cylinder radius $h/R \ll 1$.

The wall thickness is small compared to the wave-length of the sound within the wall $h/\lambda_c \ll 1$.

The acoustic properties of the tube material differ only slightly from those of the surrounding medium

$$\rho/\rho_c = 0(1) \qquad \lambda/\lambda_c = 0(1)$$

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With these assumptions the pressure- and velocity differences between inner and outer boundary of the tube-wall can be expressed in terms of the primary wave motion of the source at the origin. The boundary conditions on the inner and outer boundary, can be applied on the mean cylinder surface with radius R, after carrying out an analytical continuation of the velocity potential.

The remaining solution of the Helmholtz equation, satisfying given conditions at a surface of discontinuity is straightforward and well-known. An asymptotic expansion of this solution at large distances from the origin can be made.

The motion of the cylinder wall.

The problem is rotatory symmetric around the x-axis. Polar coordinates x, r, θ are introduced. Due to the assumption, the thickness of the cylinder wall is small compared to the wave length of sound within the wall ($h\,k_c \ll 1$), the pressure within the wall will be equal to the mean pressure at the boundaries.

So,

$$p = \frac{p_i + p_o}{2} \tag{1}$$

where subscript i indicates the inner region and o the outer region of the cylinder surface.

Linearizing for small values of h/R, the equation of continuity becomes:

$$\rho_c h R \frac{\partial u}{\partial x} + \frac{\partial}{\partial t}(\rho_c h R) = 0 \tag{2}$$

the equation of motion in x direction:

$$\frac{\partial p}{\partial x} + \rho_c \frac{\partial u}{\partial t} = 0 \tag{3}$$

the equation of motion in radial direction:

$$p_i - p_o = \rho_c h \frac{\partial^2 R}{\partial t^2} \tag{4}$$

The sound velocity within the wall of the tube is

$$c_c = \sqrt{\frac{\partial p}{\partial \rho_c}} = \frac{\omega}{k_c} \tag{5}$$

Eliminating u from (2) and (3) and substituting (5) results into

$$\frac{1}{\rho_c}\frac{\partial^2 p}{\partial x^2} = \frac{1}{h}\frac{\partial^2 h}{\partial t^2} + \frac{1}{R}\frac{\partial^2 R}{\partial t^2} + \frac{k_c^2}{\rho_c \omega^2}\frac{\partial^2 p}{\partial t^2} \tag{6}$$

Expressing the equations (4) and (6) in terms of the velocity potential $(Rc)^{-1}\Phi(x,y,z,t) = \varphi(x,y,z)e^{-i\omega t}$ using the conditions

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial r}\frac{\Phi_o + \Phi_i}{2}$$

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial r}(\Phi_o - \Phi_i) \tag{7}$$

$$p = \frac{p_i + p_o}{2} = -\rho \frac{\partial}{\partial t} \frac{(\Phi_i + \Phi_o)}{2}$$

we get the dimensionless equations:

$$\varphi_o - \varphi_i - \frac{\rho_c}{\rho} \frac{h}{2R} \frac{\partial}{\partial \eta} (\varphi_o + \varphi_i) = 0 \qquad (8)$$

and

$$\frac{\rho_c}{\rho} \frac{\partial}{\partial \eta} \left\{ \varphi_o - \varphi_i + \frac{h}{2R} (\varphi_o + \varphi_i) \right\} + \frac{h}{2R} \left\{ \frac{\partial^2}{\partial \xi^2} (\varphi_o + \varphi_i) + k_c^2 R^2 (\varphi_o + \varphi_i) \right\} = 0 \qquad (9)$$

where $\xi = \frac{x}{R}$ and

$\eta = \frac{y}{R}$ are dimensionless polar coordinates

$\varphi_o = \varphi_o(\xi, 1 + \frac{1}{2} \frac{h}{R})$ and

$\varphi_i = \varphi_i(\xi, 1 - \frac{1}{2} \frac{h}{R})$ is the velovity potential

on the outer- respectively inner cylinder surface.

By an analytical continuation of φ_o and φ_i on the mean cylinder surface $\eta = 1 \quad \xi \leqslant \frac{1}{R}$, we can write

$$\varphi_o(\xi, 1 + \tfrac{1}{2} \tfrac{h}{R}) = \varphi_o(\xi, 1) + \tfrac{1}{2} \frac{h}{R} \frac{\partial}{\partial \eta} \varphi_o(\xi, 1) + 0\left(\frac{h^2}{R^2}\right) \qquad (10)$$

$$\varphi_i(\xi, 1 - \tfrac{1}{2} \tfrac{h}{R}) = \varphi_i(\xi, 1) - \tfrac{1}{2} \frac{h}{R} \frac{\partial}{\partial \eta} \varphi_i(\xi, 1) + 0\left(\frac{h^2}{R^2}\right) \qquad (11)$$

The discontinuous boundary conditions on the cylinder with radius R transform into

$$\varphi_o - \varphi_i + \frac{h}{R} \left(1 - \frac{\rho_c}{\rho}\right) \frac{\partial}{\partial \eta} \tfrac{1}{2} (\varphi_o + \varphi_i) + 0\left(\frac{h^2}{R^2}\right) = 0 \qquad (12)$$

$$\frac{\rho_c}{\rho} \frac{\partial}{\partial \eta} (\varphi_o - \varphi_i) + \frac{h}{R} \left\{ \left(1 - \frac{\rho_c}{\rho}\right) \frac{\partial^2}{\partial \xi^2} \frac{\varphi_o + \varphi_i}{2} + \left(k_c^2 R^2 - k^2 R^2 \frac{\rho_c}{\rho}\right) \frac{\varphi_o + \varphi_i}{2} \right\} + 0\left(\frac{h^2}{R^2}\right) = 0 \qquad (13)$$

where use has been made of the fact that φ satisfies

$$\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta} + k^2 R^2 \right] \varphi = 0$$

Relation (12) is Newton's second law for the motion of the cylinder wall in radial direction, while (13) represents the equation of wave motion in axial direction within the wall.

The approximate solution.

If the acoustic properties of the cylinderwall deviate only slightly from those of the surrounding medium, we can apply Born's approximation for inhomogeneous media.

As a zero order approximation the primary wave motion of the source at the origin is taken as

$$\varphi = \varphi_s = \frac{e^{ikR\sqrt{\xi^2+\eta^2}}}{\sqrt{\xi^2+\eta^2}} \tag{14}$$

The first order approximation of the boundary conditions on the cylinder surface results into

$$\varphi_o - \varphi_i = -\frac{h}{R}\left(1-\frac{\rho_c}{\rho}\right)\left(ikR - \frac{1}{\sqrt{\xi^2+1}}\right)\frac{e^{ikR\sqrt{\xi^2+1}}}{\xi^2+1} \tag{15}$$

and

$$\frac{\partial}{\partial\eta}(\varphi_o-\varphi_i) = -\frac{h}{R}\left\{\left(\frac{\rho}{\rho_c}-1\right)\frac{\partial}{\partial\xi^2}\frac{e^{ikR\sqrt{\xi^2+1}}}{\sqrt{\xi^2+1}} + \left(k_c^2R^2\frac{\rho}{\rho_c} - k^2R^2\right)\frac{e^{ikR\sqrt{\xi^2+1}}}{\sqrt{\xi^2+1}}\right\} \tag{16}$$

The solution of the wave motion in first order approximation becomes

$$\varphi(\xi,\eta) = \varphi_s + \int_{-1/R}^{1/R}\int_0^{2\pi}\left\{\sigma(\xi_o)\frac{e^{ikh}}{h/R} + \mu(\xi_o)\left(\frac{d}{d\eta_o}\frac{e^{ikh}}{h/R}\right)_{\eta_o=1}\right\}d\theta\, d\xi_o \tag{17}$$

where

$$\frac{h}{R} = \sqrt{(\xi-\xi_o)^2+\eta_o^2+\eta^2-2\eta\,\eta_o\cos(\theta-\gamma)} \tag{18}$$

is the distance between the point $P(\xi,\ \eta\cos\gamma,\ \eta\sin\gamma)$ and the point $Q(\xi_o,\ \eta_o\cos\theta,\ \eta_o\sin\theta)$ on the cylinder surface ($\eta_o = 1$).

The problem is axisymmetric, hence the solution is independent of γ. Take $\gamma = 0$.

The first part of the integral in (17) is the potential due to a source distribution on the cylinder surface with strength $\sigma(\xi_o)$

$$\sigma(\xi_o) = -\frac{1}{4\pi}\frac{\partial}{\partial\eta}(\varphi_o-\varphi_i) = \frac{1}{4\pi}\frac{h}{R}\left\{\left(\frac{\rho}{\rho_c}-1\right)\frac{\partial^2}{\partial\xi_o^2}\frac{e^{ikR\sqrt{\xi_o^2+1}}}{\sqrt{\xi_o^2+1}} + \right.$$
$$\left. + \left(k_c^2R^2\frac{\rho}{\rho_c} - k^2R^2\right)\frac{e^{ikR\sqrt{\xi_o^2+1}}}{\sqrt{\xi_o^2+1}}\right\} \tag{19}$$

The second part of the integral in (17) can be considered as the potential due to a dipole distribution on the cylinder surface with strength $\mu(\xi_o)$

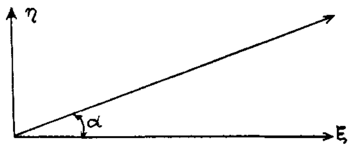
$$\mu(\xi_o) = -\frac{1}{4\pi}(\varphi_o - \varphi_i) = \frac{1}{4\pi}\,\frac{h}{R}\left(1 - \frac{\rho_c}{\rho}\right)\left(ikR - \frac{1}{\sqrt{\xi_o^2 + 1}}\right)\frac{e^{ikR\sqrt{\xi_o^2+1}}}{\xi_o^2 + 1} \tag{20}$$

We are especially interested in the radiation of sound at large distances from the origin.

It is easy to derive the asymptotic behaviour of the velocity potential expressed in equation (17) for large values of $\sqrt{\xi^2 + \eta^2}$. Introduce

$$\xi = \sqrt{\xi^2 + \eta^2}\,\cos\alpha$$

$$\eta = \sqrt{\xi^2 + \eta^2}\,\sin\alpha$$



The asymptotic expansion for $\frac{h}{R}$ reads

$$\frac{h}{R} = \sqrt{\xi^2 + \eta^2}\left[1 - 2\,\frac{\xi_o \cos\alpha + \eta_o \sin\alpha\,\cos\theta}{\sqrt{\xi^2 + \eta^2}} + \frac{\xi_o^2 + \eta_o^2}{\xi^2 + \eta^2}\right]^{\frac{1}{2}}$$

$$\approx \sqrt{\xi^2 + \eta^2}\left[1 - \frac{\xi_o \cos\alpha + \eta_o \sin\alpha\,\cos\theta}{\sqrt{\xi^2 + \eta^2}} + 0(\xi^2 + \eta^2)^{-1}\right]$$

and

$$\frac{e^{ikh}}{h/R} = \frac{e^{ikR\sqrt{\xi^2+\eta^2}}}{\sqrt{\xi^2 + \eta^2}}\; e^{-ikR(\xi_o \cos\alpha + \eta_o \sin\alpha\cos\theta)}\left[1 + 0(\xi^2 + \eta^2)^{-\frac{1}{2}}\right] \tag{21}$$

The potential at the point $P(\xi,\eta)$ becomes

$$\varphi(\xi,\eta) = \frac{e^{ikR\sqrt{\xi^2+\eta^2}}}{\sqrt{\xi^2 + \eta^2}}\left[1 + 2\pi\int_{-1/R}^{1/R} e^{-ikR\,\xi_o\cos\alpha}\left\{\sigma(\xi_o)J_o(kR\,\sin\alpha) + \right.\right.$$

$$\left.\left. + \mu(\xi_o)kR\,\sin\alpha\,J_1(kR\,\sin\alpha)\right\} d\xi_o\right] \tag{22}$$

Substituting the expressions (19) and (20) for $\sigma(\xi_o)$ and $\mu(\xi_o)$ and evaluating the integral gives the final result

$$\frac{\varphi(\xi,\eta)\cdot\sqrt{\xi^2 + \eta^2}}{e^{ikR\sqrt{\xi^2+\eta^2}}} = 1 + \frac{h}{R}J_o(kR\,\sin\alpha)\left[\left(\frac{\rho}{\rho_c} - 1\right)\cos(kl\,\cos\alpha)\frac{R^2}{l^2 + R^2}\,\cdot\right.$$

$$\cdot\left(ikl - \frac{1}{\sqrt{l^2 + R^2}}\right)e^{ik\sqrt{l^2+R^2}} + \left(\frac{\rho}{\rho_c} - 1\right)\sin(kl\,\cos\alpha)kR\,\cos\alpha\,\frac{R}{\sqrt{l^2 + R^2}}\,e^{ik\sqrt{l^2+R^2}} +$$

$$\left. + \frac{1}{2}\left\{(k_c^2R^2 - k^2R^2\cos^2\alpha)\frac{\rho}{\rho_c} - k^2R^2\sin^2\alpha\right\}\int_{-1/R}^{1/R}\frac{e^{ikR(\sqrt{\xi_o^2+1} - \xi_o\cos\alpha)}}{\sqrt{\xi_o^2 + 1}}\,d\xi_o\right] +$$

$$+ \frac{h}{2R}\,kR\,\sin\alpha\,J_1(kR\,\sin\alpha)\left(1 - \frac{\rho_c}{\rho}\right)\int_{-1/R}^{1/R}\frac{e^{ikR(\sqrt{\xi_o^2+1} - \xi_o\cos\alpha)}}{\xi_o^2 + 1}\left(ikR - \frac{1}{\sqrt{\xi_o^2 + 1}}\right)d\xi_o \tag{23}$$

Some particular solutions of the problem.

The general solution as expressed by equation 23 reduces to a more simple result for the following particular cases.

Case 1.

The frequency of the sound emitted by the point source tends to zero. Expanding equation 23 in power series of the constant kR results into.

$$\varphi(\xi,\eta) = (\xi^2+\eta^2)^{-\frac{1}{2}} e^{ikR\sqrt{\xi^2+\eta^2}} \left\{1 + \frac{h}{R}\left(1 - \frac{\rho}{\rho_c}\right) \frac{R^2 1}{(1^2+R^2)^{3/2}} + \frac{h}{R}\, 0(k^2R^2)\right\}$$

So the amplitude of the perturbed wave motion is inversely proportional to the distance $\sqrt{\xi^2+\eta^2}$, and to the square of the wave-length λ.

Case 2.

The frequency of the sound wave tends to infinity.

Still satisfying the conditions, $hk_c \ll 1$ and k/k_c is order unity, involves that

$$h/R \ll (kR)^{-1} \text{ approaches zero.}$$

For large values of the constant kR an asymptotic approximation of the integrals in expression 23 can be obtained by the application of the method of stationary phase.

The mean contribution to the integral

$$I_1 = \int_{-1(R)}^{1/R} (\xi^2+1)^{-\frac{1}{2}} e^{ikR(\sqrt{\xi^2+1} - \xi\cos\alpha)}\, d\xi$$

arises from the neighbourhood of that point ξ_o of the integration interval at which the phase of the oscillatory part of the integrand is stationary.

The value of ξ_o is determined by the relation

$$\frac{d}{d\xi}\left(\sqrt{\xi^2+1} - \xi\cos\alpha\right) = 0$$

so $\xi_o = \text{cotg}\,\alpha$.

The integral I_1 is of order $(kR)^{-\frac{1}{2}}$ when the stationary point ξ_o lies in the interior of the interval $-1/R \leqslant \xi \leqslant 1/R$ and of order $(kR)^{-1}$ when ξ_o lies outside the interval.

The approximate solution is

$$I_1 = \sqrt{\frac{2\pi}{kR\sin\alpha}}\, e^{i(kR\sin\alpha + \frac{\pi}{4})} + 0(kR)^{-3/2} \quad \text{if} \quad \text{cotg}\,\alpha \leqslant \frac{1}{R}$$

$$= 0(kR)^{-1} \quad \text{if} \quad \text{cotg}\,\alpha > \frac{1}{R}$$

Likewise the second integral in equation 23 becomes

$$I_2 = \int_{-1/R}^{1/R} \left\{(\xi^2+1)^{-1} ikR - (\xi^2+1)^{-3/2}\right\} e^{ikR(\sqrt{\xi^2+1} - \xi\cos\alpha)}\, d\xi =$$

$$= i\sqrt{2\pi kR\sin\alpha}\, e^{i(kR\sin\alpha + \frac{\pi}{4})} + 0(kR)^{-\frac{1}{2}} \quad \text{if} \quad \text{cotg}\,\alpha \leqslant \frac{1}{R}$$

$$= 0(1) \quad \text{if} \quad \text{cotg}\,\alpha > \frac{1}{R}$$

Using the asymptotic expressions of the besselfunctions $J_0(kR\sin\alpha)$ and $J_1(kR\sin\alpha)$ for large values of kR the final result becomes

$$|\varphi(\xi,\eta)| = (\xi^2+\eta^2)^{-\frac{1}{2}}\left[1+\frac{h}{R}\left[\tfrac{1}{2}kR\sin\alpha\cos(2kR\sin\alpha)\cdot\left\{\left(\frac{k_c^2-k^2}{k^2\sin^2\alpha}\right)\frac{\rho}{\rho_c}+\frac{\rho}{\rho_c}-\frac{\rho_c}{\rho}\right\}+0(kR)^{+\frac{1}{2}}\right]\right] \quad \text{if} \quad \text{cotg}\,\alpha \leqslant \frac{1}{R}$$

$$= (\xi^2+\eta^2)^{-\frac{1}{2}}\left[1+\frac{h}{R}\,0(kR)^{+\frac{1}{2}}\right] \quad \text{if} \quad \text{cotg}\,\alpha > \frac{1}{R}$$

From the above it can be concluded that the perturbation of the primary wave motion at large distances from the source is greater in the "shadow region" $\text{cotg}\,\alpha \leqslant \frac{1}{R}$ than it is in the region $\text{cotg}\,\alpha > \frac{1}{R}$. Whether the intensity of the radiated sound in the shadow region is increased or decreased depends on the sign of the term $\left(\frac{k_c^2-k^2}{k^2\sin^2\alpha}\frac{\rho}{\rho_c}+\frac{\rho}{\rho_c}-\frac{\rho_c}{\rho}\right)$.

Case 3.
When the length of the cylindrical tube tends to infinity the integrals in equation 23 can be written as:

$$\lim_{1/R\to\infty}\int_{-1/R}^{1/R}(\xi^2+1)^{-\frac{1}{2}}\,e^{ikR(\sqrt{\xi^2+1}-\xi\cos\alpha)}\,d\xi =$$

$$= \lim_{1/R\to\infty}2\int_{0}^{1/R}(\xi^2+1)^{\frac{1}{2}}\,e^{ikR\sqrt{\xi^2+1}}\cos(\xi kR\cos\alpha)d\xi = i\pi H_0^{(1)}(kR\sin\alpha)$$

and

$$\lim_{1/R\to\infty}\int_{-1/R}^{1/R}e^{ikR(\sqrt{\xi^2+1}-\xi\cos\alpha)}\left\{ikR(\xi^2+1)^{-1}-(\xi^2+1)^{-3/2}\right\}d\xi =$$

$$= -i\pi kR\sin\alpha\, H_1^{(1)}(kR\sin\alpha)$$

The velocity potential at large distances from the origin for an infinite long cylinder becomes:

$$\frac{\varphi(\xi,\eta)\sqrt{\xi^2+\eta^2}}{e^{ikR\sqrt{\xi^2+\eta^2}}} = 1+i\pi\frac{h}{2R}\left[\left\{(k_c^2R^2-k^2R^2\cos^2\alpha)\frac{\rho}{\rho_c}-k^2R^2\sin^2\alpha\right\}\cdot\right.$$

$$\left. J_0(kR\sin\alpha)H_0^{(1)}(kR\sin\alpha)-k^2R^2\sin^2\alpha\left(1-\frac{\rho_c}{\rho}\right)J_1(kR\sin\alpha)H_1^{(1)}(kR\sin\alpha)\right]$$

The same result has been obtained by P. le Grand [1], who used a Fourier transform technique.

Numerical results.
Equation 23 is numerically evaluated for the special case $h/R = 0.1$ $kR = 1.5$ $\alpha = \pi/2$ $1/R = 1$ $\rho/\rho_c = 1.2$ $k/k_c = 0.8$.
Each one of these quantities is varied, leaving the other quantities fixed.
The reduction of sound intensity far away from the source is expressed

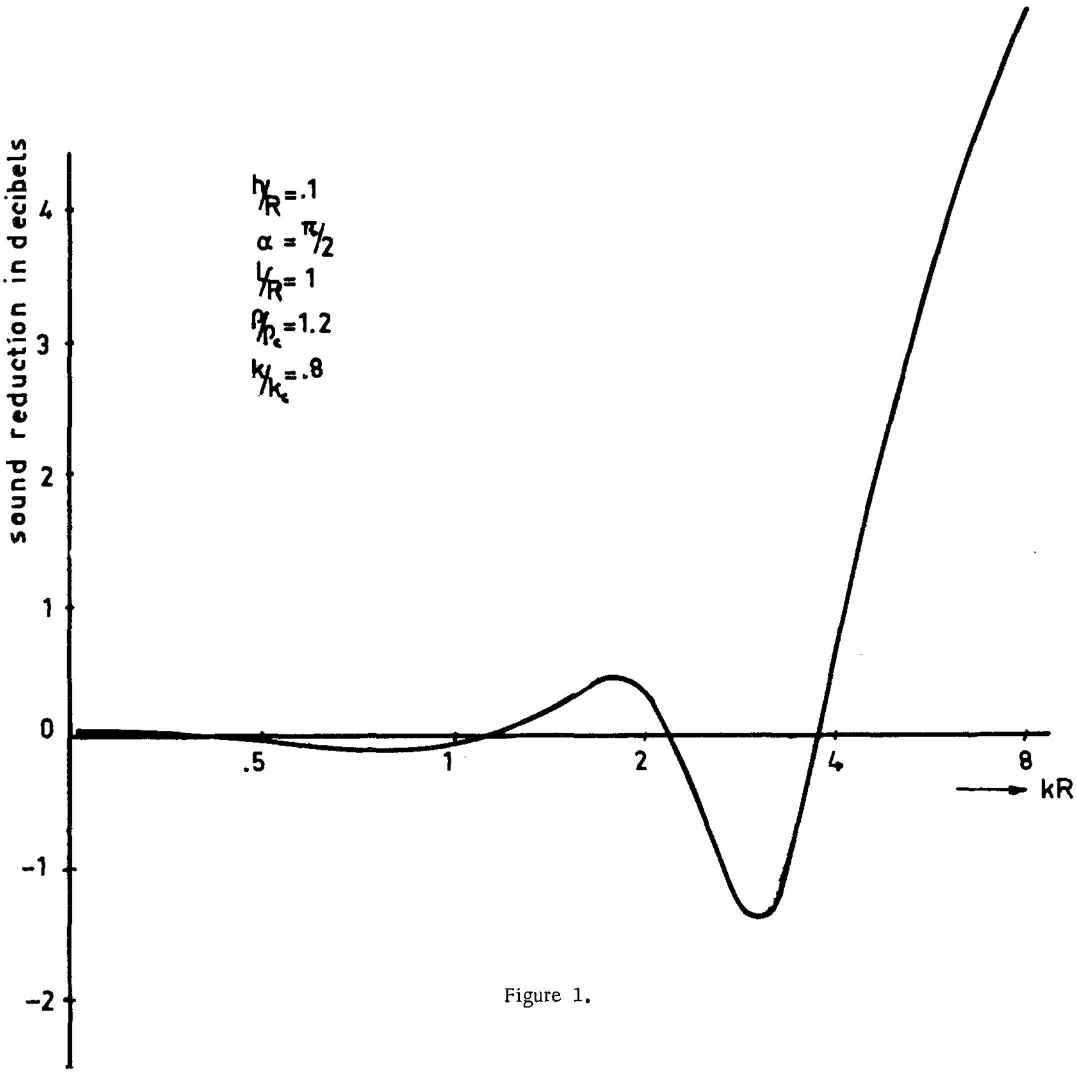


Figure 1.

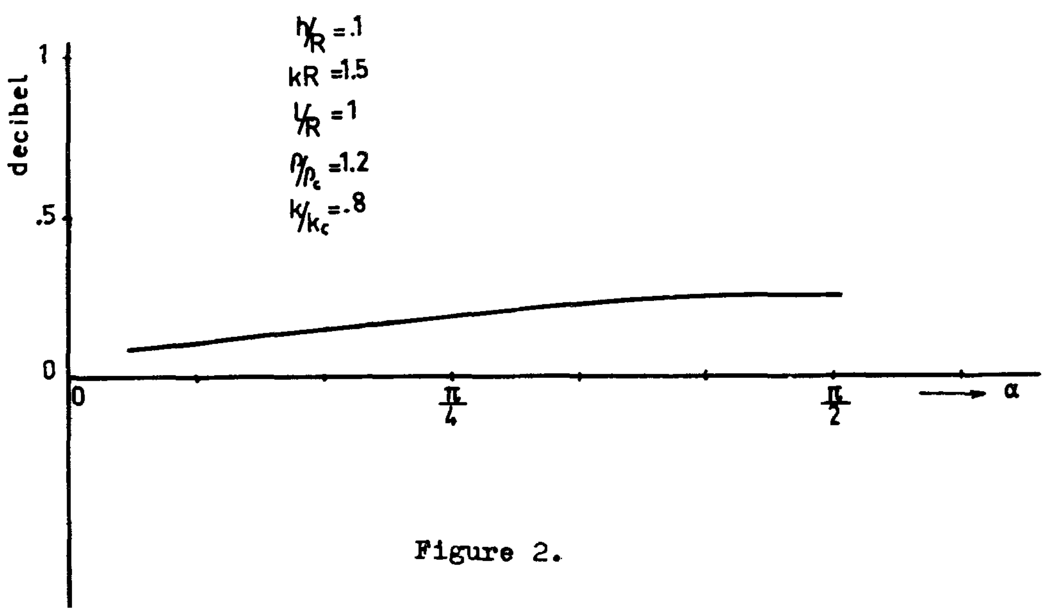


Figure 2.

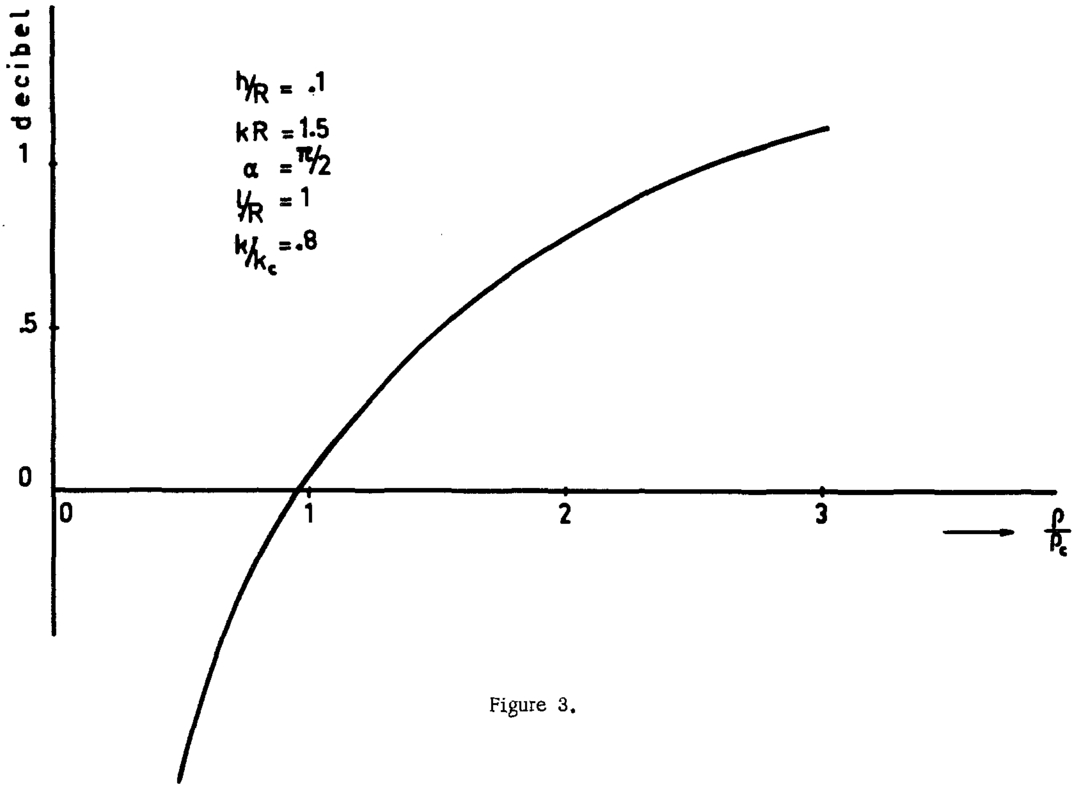


Figure 3.

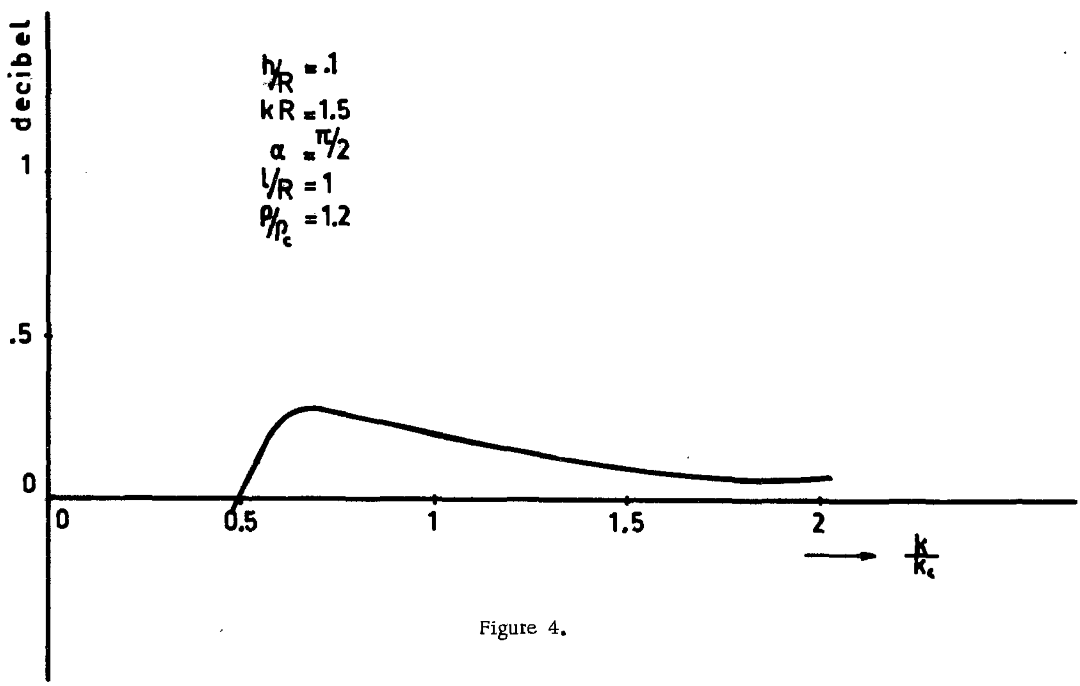


Figure 4.

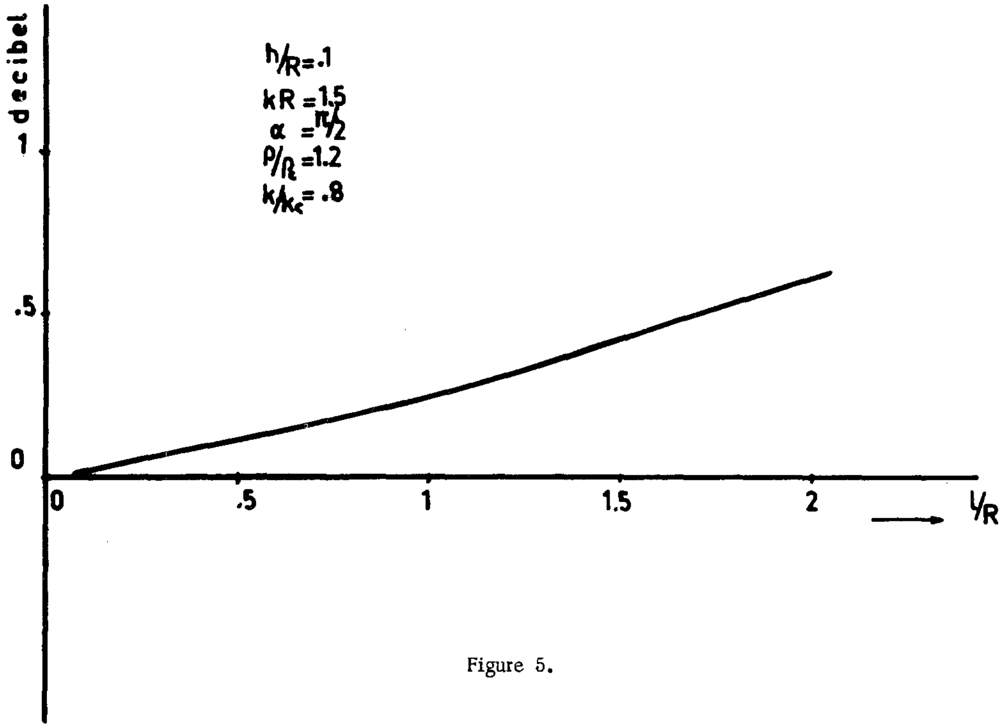


Figure 5.

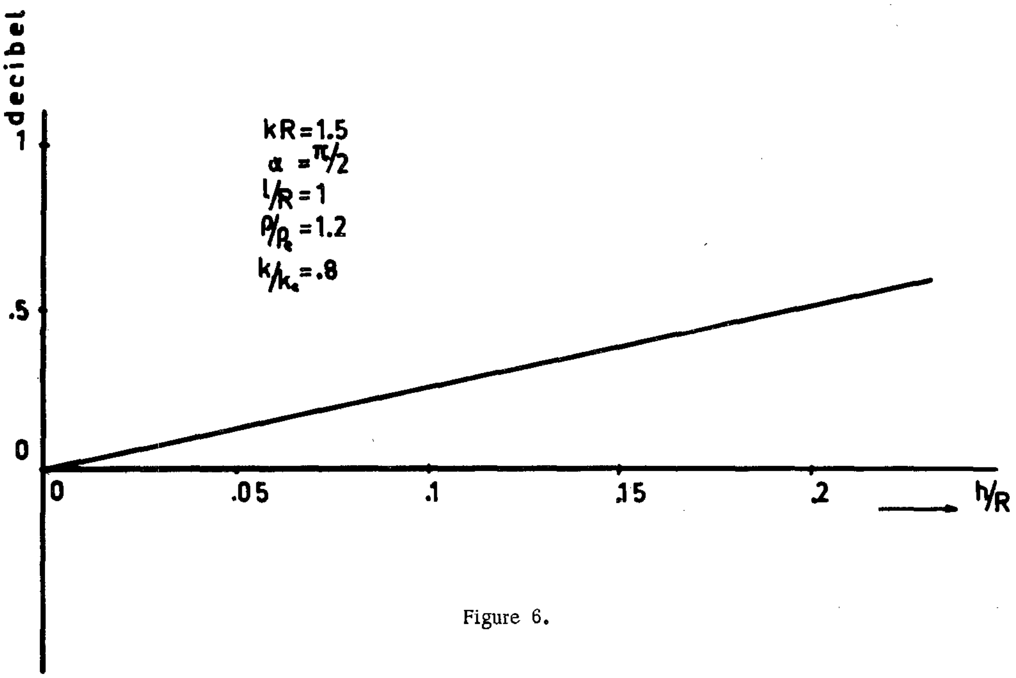


Figure 6.

in decibels according to the definition equation

$$D = -20\ {}^{10}\log \left| \varphi(\xi,\eta)\sqrt{\xi^2+\eta^2} \right|$$

Fig. 1 till 6 show the sound reduction due to the effect of the cylindrical screen as a function of the various parameters.

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